Stability analysis of some classes of input-affine nonlinear systems with aperiodic sampled-data control

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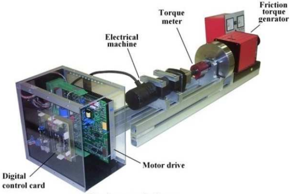
Ecole Centrale de Lyon 14/10/2015

Research activities

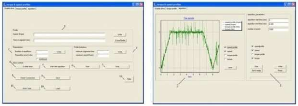
- *2006-2009: Research engineer*
  Higher Institute for Applied Science and Technology (HIAST)
  Control of power electronics and electrical machines

  - *Active Power Filters*
  - *Digital control of DC motors*

    Framework: european project TEMPUS "FINSI"
  - *Dip-coating machine*



Hardware platform



Research activities

- *2010: Master thesis*
  École Centrale de Nantes/IRCCyN
  Supervisor: J.-J. Loiseau (DR CNRS)

"Positive systems and robustness"

Research activities

- *2011 - 2014:* ***École Centrale de Lille*** *-* ***LAGIS UMR CNRS 8219***
  Work group ***SyNeR***
  - *Thèse de Doctorat*

    " Contribution to the control of nonlinear systems under aperiodic sampling"

- *Supervisors:*
  Jean-Pierre Richard (Prof. EC Lille)
  Laurentiu Hetel (CR CNRS)
  Françoise Lamnabhi-Lagarrigue (DR CNRS)

  CNRS european project HYCON2

Research activities

- *April 2014 - November 2014: Post-doc*
  École des Mines de Douai
  M. Petreczky (Assistant Professor)

  ESTIREZ: regional project Nord-Pas-de-Calais
  LAGIS, INRIA, Ecole des Mines de Douai

Research activities

- *December 2014 - August 2015: Post-doc*
  Railenium/LAMIH - UMR CNRS 8201
  Project ECOVIGIDRIV

  " Predictive control of hybrid systems"

  Simon Enjalbert (Assistant Professor, LAMIH)
  Sébastien Lefebvre (Project manager, Railenium)

Research activities

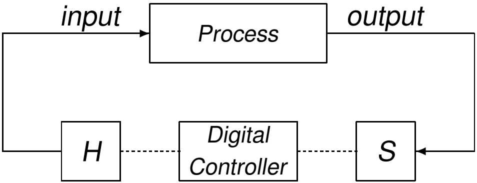
- *September 2015:* *ATER*
  École Centrale de Lyon - Ampère UMR 5005

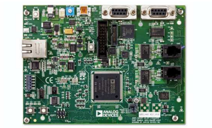
  "Application of robust control to the design of electronic circuits"

  Gerard Scorletti (Professeur, École Centrale de Lyon)
  Anton Korniienko (Assistant Professor, École Centrale de Lyon)

Stability analysis of some classes of input-affine nonlinear systems with aperiodic sampled-data control

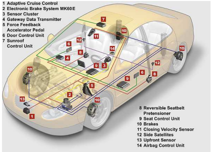
Sampled-data control systems

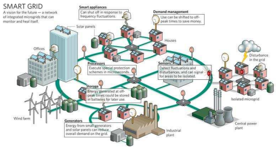




Digital/Networked Control Systems

Applications of sampled-data control systems





Challenges in sampled-data control

$$\left.\begin{array}{l}\textbf{\textit{Processor}}\text{: \textit{limited calculation power}}\\ \textbf{\textit{Network}}\text{: \textit{finite bandwidth}}\\ \textbf{\textit{Sampler}}\text{: \textit{minimum responding time}}\end{array}\right\} \Rightarrow \text{\textit{finite number of samples per time unit}}$$

How fast SHOULD we sample? $\leftrightarrow$ *How fast CAN we sample?*

Challenges in sampled-data control

Sampler clock: *jitter*
Network: *packet dropouts*
Scheduling: *interaction between algorithms*
Real-time computing: *microprocessor latency*

$\Rightarrow$ *sampling is not necessarily periodic*

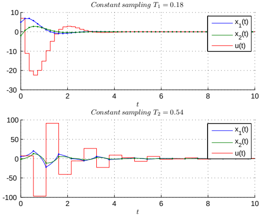
How to ensure robustness with respect to asynchronous sampling?

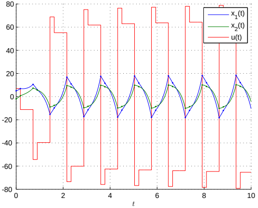
Challenges in sampled-data control

Aperiodic sampling may cause instability

$$\dot{x}(t) = A_0 x(t) + B_0 u(t), \qquad u(t) = Kx(t_k)$$

$$A_0 = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}, \quad B_0 = \begin{bmatrix} 1 \\ 0.6 \end{bmatrix}, \quad K = \begin{bmatrix} -1 & -6 \end{bmatrix}.$$

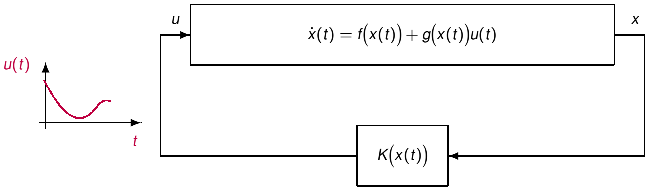




Periodic sampling $T_1 = 0.18$, $T_2 = 0.54$ *Aperiodic sampling* $T_1 \to T_2 \to T_1 \cdots$

Problem under study

Continuous-time controller

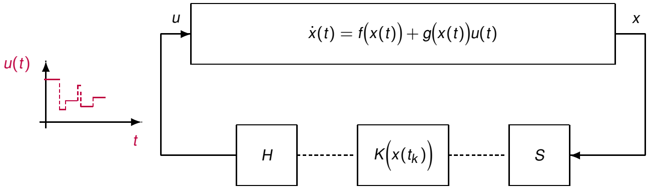


$$u(t) = K\big(x(t)\big)$$

where $f(\cdot)$, $g(\cdot)$ and $K(\cdot)$ are continuously differentiable functions on a neighborhood of the origin $x = 0$ denoted $\mathcal{D}$.

Problem under study

Digital implementation under asynchronous sampling (emulation approach)



$$u(t) = K\Big(x(t_k)\Big), \quad \forall t \in [t_k, t_{k+1}), \qquad 0 < t_{k+1} - t_k \leq \underbrace{\bar{h}}_{MASP}, \quad \forall k \in \mathbb{N}.$$

Find stability criteria for nonlinear sampled-data control systems, which provide a computable estimate of the Maximum Allowable Sampling Period (MASP).

Existing results

The linear time-invariant case: Realistic model?

- *Input delay approach (Fridman et al 2004), (Fridman 2010), (Michiels 2005)*
- *Robust control based analysis (Mirkin 2007), (Fujioka 2009)*
- *Impulsive modelling (Naghshtabrizi 2008)*
- *Discrete-time approaches & convex embedding (Hetel, Daafouz et al 2007)*
- *Sum of squares (Seuret 2011)*

The nonlinear case: Constructive?

- *Input delay approach (Mazenc et al 2013)*
- *Hybrid system modelling (Nešić et al 2009), (Burlion et al 2006)*
- *Single/vector Lyapunov functions (Karafyllis et al 2007)*
- *L_p stability (Zaccarian et al 2003)*

Dissipativity-based representation

Equivalent model

Closed-loop system

$$\dot{x}(t) = f\big(x(t)\big) + g\big(x(t)\big)K\big(x(t_k)\big)$$

$$\dot{x}(t) = \underbrace{f\big(x(t)\big) + g\big(x(t)\big)K\big(x(t)\big)}_{f_n(x(t))} + \underbrace{g\big(x(t)\big)}_{g_n(x(t))}\underbrace{\Big(K\big(x(t_k)\big) - K\big(x(t)\big)\Big)}_{w(t)}$$

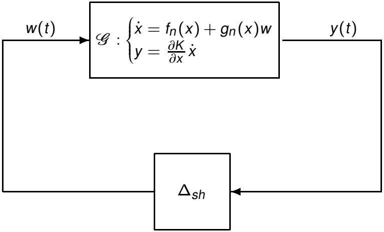
The system can be represented by the interconnection of:

$$\mathscr{G} : \begin{cases} \dot{x}(t) = f_n\big(x(t)\big) + g_n\big(x(t)\big)w(t) \\ y(t) = \frac{\partial K}{\partial x}\dot{x}(t) \end{cases}$$

with the operator $\Delta_{sh} : y \to w$ defined by:

$$w(t) = (\Delta_{sh}\, y)(t) = -\int_{t_k}^{t} y(\tau)d\tau$$

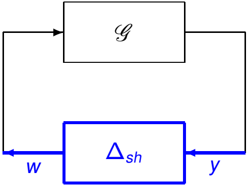
Dissipativity-based representation



$$\mathscr{G}: \begin{cases} \dot{x}(t) = & f_n(x(t)) + g_n(x(t))w(t) \\ y(t) = & \frac{\partial K}{\partial x}\dot{x} \end{cases}$$

$$(\Delta_{sh}y)(t) := -\int_{t_k}^{t} y(\tau)d\tau.$$

Properties of the operator



L_2-induced norm (Mirkin 2007):

$$\frac{\|w\|}{\|y\|} \leq \delta_0 := \frac{2}{\pi}\overline{h} \quad i.e. \int_0^\infty w^T(\tau)w(\tau)d\tau \leq \delta_0^2 \int_0^\infty y^T(\tau)y(\tau)d\tau.$$

Passivity (Fujioka 2009):

$$\langle \Delta_{sh}y, y\rangle = \int_0^\infty y^T(\tau)\,(\Delta_{sh}y)(\tau)d\tau \leq 0.$$

Properties of the operator

Boundedness property

For all $y \in L_2[t_k, t_{k+1})$ *and* $0 < X^* = X \in \mathbb{R}^{n\times n}$:

$$\int_{t_k}^{t} (\Delta_{sh} y)^* X (\Delta_{sh} y)\, d\tau - \delta_0^2 \int_{t_k}^{t} y^* X y\, d\tau \leq 0, \quad \forall t \in [t_k, t_{k+1})$$

Passivity property

For all $y \in L_2[t_k, t_{k+1})$ *and* $0 \leq Y^* = Y \in \mathbb{R}^{n\times n}$:

$$\int_{t_k}^{t} (\Delta_{sh} y)^* Y y\, d\tau + \int_{t_k}^{t} y^* Y (\Delta_{sh} y)\, d\tau \leq 0, \quad \forall t \in [t_k, t_{k+1})$$

$$\Rightarrow \int_{t_k}^{t} \underbrace{\begin{bmatrix} y \\ w \end{bmatrix}^T \begin{bmatrix} -\delta_0^2 X & Y \\ Y & X \end{bmatrix} \begin{bmatrix} y \\ w \end{bmatrix}}_{\mathcal{S}(y,w)} d\tau \leq 0, \quad \forall t \in [t_k, t_{k+1})$$

Main result

Theorem

Suppose there exist $\alpha > 0$, $0 \leq X^T = X \in \mathbb{R}^{m\times m}$, $0 \leq Y^T = Y \in \mathbb{R}^{m\times m}$, *and a differentiable positive definite function* $V : \mathcal{D} \to \mathbb{R}^+$, *which radially unbounded, such that*

$$\frac{\partial V}{\partial x}\Big(f_n(x) + g_n(x)w\Big) + \alpha V(x) \leq \Bigg(-\delta_0^2 \left\| \frac{\partial K}{\partial x}\Big(f_n(x) + g_n(x)w\Big) \right\|_X^2 + \|w\|_X^2 + 2\langle \frac{\partial K}{\partial x}\Big(f_n(x) + g_n(x)w\Big), w\rangle_Y \Bigg) e^{-\alpha \bar{h} l}, \forall l \in \{0, 1\}, \tag{1}$$

for all $x \in \mathcal{D}$ *and* $w \in \mathbb{R}^m$. *Then the equilibrium* $x = 0$ *of the sampled-data control system is locally uniformly asymptotically stable, and the decay rate of the function* $V\big(x(t)\big)$ *is* α.

$\langle u, v\rangle_M = u^T M v$, $\|u\|_M = \sqrt{\langle u, u\rangle_M}$.

Main result

Main idea:

Local Lipschitz continuity of the vector field

+

Equation (1)

$\Downarrow$

$$\dot{V}\big(x(t)\big) + \alpha V\big(x(t)\big) \quad \leq \quad e^{-\alpha(t-t_k)}\mathcal{S}\big(y(t), w(t)\big), \ \forall t \in [t_k, t_{k+1}), \tag{2}$$

Then, using the properties of Δ_{sh} *we show that*

$$V\big(x(t)\big) \leq e^{-\alpha(t-t_0)} V\big(x(t_0)\big), \quad \forall t \geq t_0, \quad \forall x_0 \in \mathcal{L}_{c^*}, \tag{3}$$

where $\mathcal{L}_{c^*}$ *is the maximal sub-level set of* V *contained in* D*.*

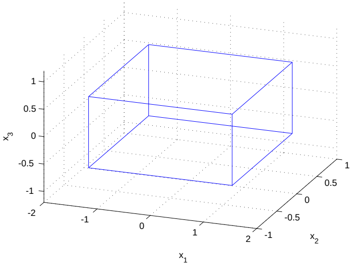
Case study 1: polytopic systems

Consider the problem for the following case

$$\dot{x}(t) = A(x(t))x(t) + B(x(t))u(t), \qquad u(t) = Fx(t)$$

where $A(\cdot) \in \mathbb{R}^{n\times n}$, $B(\cdot) \in \mathbb{R}^{n\times m}$ *are continuously differentiable functions*

$$[A(x), B(x)] \quad \in \quad conv\{[A_1, B_1], [A_2, B_2], \cdots, [A_p, B_p], \}, \quad \forall x \in \mathcal{D},$$



Case study 1: polytopic systems

Corollary

Suppose there exist symmetric positive definite matrices X, $Y \in \mathbb{R}^{m\times m}$, $P \in \mathbb{R}^{n\times n}$, matrices P_2, $P_3 \in \mathbb{R}^{n\times n}$, and a scalar $\alpha > 0$ such that the following LMIs are feasible

$$\begin{bmatrix} \alpha P + (A_i + B_i F)^T P_2 + P_2^T (A_i + B_i F) & P - P_2{}^T + (A_i + B_i F)^T P_3 & P_2{}^T B_i \\ * & -P_3 - P_3^T + \delta_0^2 e^{-\alpha \bar{h} l} F^T X F & P_3{}^T B_i - e^{-\alpha \bar{h} l} F^T Y \\ * & * & -e^{-\alpha \bar{h} l} X \end{bmatrix} < 0,$$
$$\forall i \in \{1, 2, \cdots, p\}, \quad \forall l \in \{0, 1\}. \tag{4}$$

Then the equilibrium $x = 0$ of the sampled-data control system is locally asymptotically stable. An estimate of the domain of attraction is

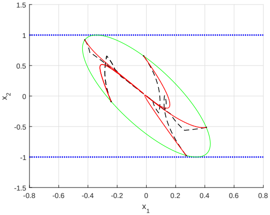
$$\mathcal{E}_{c^*} := \{x \in \mathbb{R}^n : x^T P x \le c^*\}, \quad c^* = \max_{\mathcal{E}_c \subset \mathcal{D}} c. \tag{5}$$

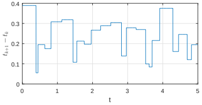
Case study 1: polytopic systems

Example:

$$\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\end{pmatrix} = \begin{pmatrix}1+0.1x_2 & 1\\ 0 & 0.1sin(x_1)\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix} + \begin{pmatrix}0\\ 1\end{pmatrix}u, \qquad F = [-6 \;\; -4].$$

$$\overline{h} = 0.389 \; and \; \alpha = 0.01$$





Case study 2: polynomial systems

Consider the problem for the case where $f(\cdot)$, $g(\cdot)$ and $K(\cdot)$ polynomials. I this case the nominal system is also polynomial:

$$\mathscr{G} : \begin{cases} \dot{x} = F(x, w), \\ y = G(x, w). \end{cases}$$

Verification of $p(\xi) \geq 0$ is a difficult problem! $\rightarrow$ simplification using SOS

A multivariate polynomial $p(\xi) \in \mathbb{R}[\xi]$ is said to be a sum of squares (SOS) (Papachristodoulou 2005) if there exist $p_i(\xi) \in \mathbb{R}[\xi]$, $i \in \{1, \ldots, M\}$, such that $p(\xi) = \sum_{i=1}^{M} p_i^2(\xi)$.

Case study 2: polynomial systems

Corollary

Suppose that there exist a polynomial function $V(x) \in \mathbb{R}[x]$ *of degree* $2d$, $0 < X^T = X \in \mathbb{R}^{m \times m}$, $0 \leq Y^T = Y \in \mathbb{R}^{m \times m}$ *and* $\alpha > 0$ *such that the following polynomials are SOS*

$$\hat{V}(x) = V(x) - \varphi(x),$$
$$\rho_1(\xi) = -\frac{\partial V}{\partial x} F(x,w) - \alpha V(x) + \Big[-(\frac{2}{\pi}\bar{h})^2 G^T(x,w) X G(x,w) + 2G^T(x,w) Y w + w^T X w \Big],$$
$$\rho_2(\xi) = -\frac{\partial V}{\partial x} F(x,w) - \alpha V(x) + \Big[-(\frac{2}{\pi}\bar{h})^2 G^T(x,w) X G(x,w) + 2G^T(x,w) Y w + w^T X w \Big] e^{-\alpha \bar{h}}.$$

with $\xi = (x, w)$ *and* $\varphi(x)$ *a positive definite polynomial. Then, the equilibrium* $x = 0$ *of the sampled-data system is globally uniformly asymptotically stable.*

Case study 2: polynomial systems

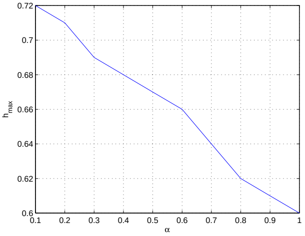
Example:

Consider the example from (Nešić et al 2009)

$$\dot{x} = dx^2 - x^3 + u, \quad u = K(x) = -2x, \quad \text{with } |d| \leq 1.$$

	Corollary	*(Nešić et al 2009)*	*(Karafyllis et al 2007)*
$\overline{h}$	0.72	0.368	0.1428

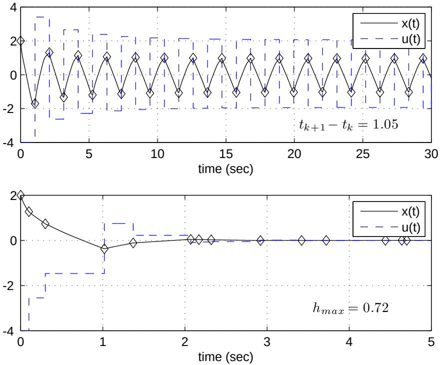
Trade-off between the decay rate α and the MASP:



Case study 2: polynomial systems

Example:

State evolution for the sampled-data system:



Summary

- *Address a quite general class of systems thanks to exponential dissipativity.*
- *Sufficient conditions for the stability of nonlinear sampled-data systems, which are affine in the control.*
- *The results are numerically tractable for the cases of polytopic and polynomial systems .*
- *Perspectives: include other network-imposed imperfections, decrease the conservatism, consider controlled sampling.*

Publications

List of publications:

- *Book chapters:*
  - *1 chapter in **Springer** accepted*
- *International journals:*
  - *1 article **Automatica** published*
  - *1 article **Automatica** submitted*
  - *1 article **WASET** published*
- *National journals (francophone)*
  - *1 article **JESA** published*
- *Conferences:*
  - ***ADHS 12**, **CDC 12**, **ECC 13**, **ECC 14**, **JDMACS 13***
- *Deliverables:*
  - *1 Deliverable 2.1.3. HYCON2*
  - *1 Deliverable ESTIREZ*

Thank you for your attention